The second Yamabe invariant

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THE SECOND YAMABE INVARIANT

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Abstract. Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. We define the second Yamabe invariant as the infimum of the second eigenvalue of the Yamabe operator over the metrics conformal to g and of volume 1. We study when it is attained. As an application, we find nodal solutions of the Yamabe equation.

April 10, 2007

MSC 2000: 53A30, 35J60(Primary) 35P30, 58J50, 58C40 (Secondary)

Contents

1. Introduction

Let (M,g) be an n-dimensional compact Riemannian manifold ($n \geq 3$). In [Yam60] Yamabe attempted to show that there is a metric $\tilde{g}$ conformal to g such that the scalar curvature $S_{\tilde{g}}$ of $\tilde{g}$ is constant. However, Trudinger [Tru68] realized that Yamabe's proof contained a serious gap. The problem is now solved, but it took a very long time to find the good approach. The problem of finding a metric $\tilde{g}$ with constant scalar curvature in the conformal class $[g]$ is called the Yamabe problem. The first step towards a rigorous solution of this problem was achieved by Trudinger [Tru68] who was able to repair the gap of Yamabe's article in the case that the scalar curvature of g is non-positive. Eight years later, Aubin [Aub76] solved the problem for arbitrary non locally conformally flat manifolds of dimension $n \geq 6$. The problem was completely solved another eight years later in an article of Schoen [Sch84] in which the proof was reduced to the positive-mass theorem which had previously been proved by Schoen and Yau [SY79, SY88]. The reader can refer to [LP87], [Aub76] or [Heb97] for more information on this subject. The method to solve

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the Yamabe problem was the following. Let $u \in C^\infty(M)$, $u > 0$ be a smooth function and $\tilde{g} = u^{N-2}g$ where $N = \frac{2n}{n-2}$. Then, multiplying u by a constant, the following equation is satisfied:

$$L_g(u) = S_{\tilde{g}}|u|^{N-2}u.$$

where

$$L_g = c_n\Delta_g + S_g = \frac{4(n-1)}{n-2}\Delta_g + S_g$$

is called the Yamabe operator. As a consequence, solving the Yamabe problem is equivalent to finding a positive smooth solution u of

$$L_g(u) = C_0|u|^{N-2}u. \tag{1}$$

where C_0 is a constant. In order to obtain solutions of this equation Yamabe defined the quantity

$$\mu(M,g) = \inf_{u\neq 0, u\in C^\infty(M)} Y(u)$$

where

$$Y(u) = \frac{\int_M c_n|\nabla u|^2 + S_g u^2\, dv_g}{\left(\int_M |u|^N\, dv_g\right)^{\frac{2}{N}}}.$$

Nowadays, $\mu(M,g)$ is called the *Yamabe invariant*, and Y the *Yamabe functional*. Writing the Euler-Lagrange equation associated to Y, we see that there exists a one to one correspondence between critical points of Y and solutions of equation (1). In particular, if u is a positive smooth function such that $Y(u) = \mu(M,g)$, then u is a solution of (1) and $\tilde{g} = u^{N-2}g$ is the desired metric of constant scalar curvature. The key point of the resolution of the Yamabe problem is the following theorem due to Aubin [Aub76]. In the theorem and in the whole article, $\mathbb{S}^N$ will always denote the sphere S^n with the standard Riemannian structure.

Theorem 1.1. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. If $\mu(M,g) < \mu(\mathbb{S}^n)$, then there exists a positive smooth function u such that $Y(u) = \mu(M,g)$.*

This strict inequality is used to show that a minimizing sequence does not concentrate in any point. Aubin [Aub76] and Schoen [Sch84] proved the following.

Theorem 1.2. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Then $\mu(M,g) \leq \mu(\mathbb{S}^n) = n(n-1)\omega_n^{\frac{2}{n}}$ where ω_n stands for the volume of the standard sphere S^n. Moreover, we have equality in this inequality if and only if (M,g) is conformally diffeomorphic to the sphere.*

These theorems solves the Yamabe problem.

In this paper, we introduce and study an invariant that we will call the *second Yamabe invariant.* It is well known that the operator L_g has discrete spectrum

$$\mathrm{Spec}(L_g) = \{\lambda_1(g), \lambda_2(g), \cdots\}$$

where the eigenvalues

$$\lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \cdots \leq \lambda_k(g) \cdots \to +\infty$$

appear with their multiplicities. The variational characterization of $\lambda_1(g)$ is given by

$$\lambda_1(g) = \inf_{u\neq 0, u\in C^\infty(M)} \frac{\int_M c_n|\nabla u|^2 + S_g u^2\, dv_g}{\int_M |u|^2\, dv_g}.$$

Let $[g]$ be the conformal class of g. Assume now that the Yamabe invariant $\mu(M,g) \geq 0$. It is easy to check that

$$\mu(M,g) = \inf_{\tilde{g}\in[g]} \lambda_1(\tilde{g})\mathrm{Vol}(M,\tilde{g})^{\frac{2}{n}},$$

where $[g]$ is the conformal class of g. We then enlarge this definition.

Definition 1.3. Let $k \in \mathbb{N}^*$. Then, the k^{th} Yamabe invariant is defined by

$$\mu_k(M,g) = \inf_{\tilde{g} \in [g]} \lambda_k(\tilde{g}) \mathrm{Vol}(M,\tilde{g})^{\frac{2}{n}}.$$

With these notations, $\mu_1(M,g)$ equals to Yamabe invariant $\mu(M,g)$ in the case $\mu(M,g) \geq 0$, and $\mu_1(M,g) = -\infty$ in the case $\mu(M,g) < 0$.

The goal of this article is to study the second Yamabe invariant $\mu_2(M,g)$ for manifolds whose Yamabe invariant in the case $\mu(M,g) \geq 0$. As explained in Section 8, the most interesting case is when $\mu(M,g) > 0$. In particular, we discuss whether $\mu_2(M,g)$ is attained. This question is discussed in Subsection 5.1. In particular, Proposition 5.2 asserts that contrary to the standard Yamabe invariant, $\mu_2(M,g)$ cannot be attained by a metric if M is connected. In other words, there does not exist $\tilde{g} \in [g]$ such that $\mu_2(M,g) = \lambda_2(\tilde{g})\mathrm{Vol}(M,\tilde{g})^{\frac{2}{n}}$. In order to find minimizers, we enlarge the conformal class $[g]$ to what we call the class of *generalized metrics* conformal to g. A generalized metric is a "metric" of the form $\tilde{g} = u^{N-2}g$, where u is no longer necessarily positive and smooth, but $u \in L^N(M)$, $u \geq 0$, $u \not\equiv 0$. The definitions of $\lambda_2(\tilde{g})$ and of $\mathrm{Vol}(M,\tilde{g})$ can be extended to generalized metrics (see section 3). Then, we are able to prove the following result:

Theorem 1.4. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$ whose Yamabe invariant is non-negative. Then, $\mu_2(M,g)$ is attained by a generalized metric in the following cases:*

- $\mu_1(M,g) > 0$ *and* $\mu_2(M,g) < \left[\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n))^{\frac{n}{2}}\right]^{\frac{2}{n}}$*;*
- $\mu_1(M,g) = 0$ *and* $\mu_2(M,g) < \mu_1(\mathbb{S}^n)$

where $\mu_1(\mathbb{S}^n) = n(n-1)\omega_n^{\frac{2}{n}}$ *is the Yamabe invariant of the standard sphere.*

The result we obtain in the case $\mu_1(M,g) = 0$ is not surprising. Indeed, when $\mu_2(M,g) < \mu_1(\mathbb{S}^n)$, Aubin's methods [Aub76] can be adapted here and allow to avoid concentration of minimizing sequences. However, when $\mu_1(M,g) > 0$ and $\mu_2(M,g) < \left[\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n))^{\frac{n}{2}}\right]^{\frac{2}{n}}$, the result is much more difficult to obtain (see Subsection 6). A second result is to find explicit examples for which the assumptions of Theorem 1.4 are satisfied. The method consists in finding an appropriate couple of test functions.

Theorem 1.5. *The assumptions of Theorem 1.4 are satisfied in the following cases:*

- $\mu_1(M,g) > 0$*, (M,g) is not locally conformally flat and $n \geq 11$;*
- $\mu_1(M,g) = 0$*, (M,g) is not locally conformally flat and $n \geq 9$.*

One of our motivations is to find solutions of the Yamabe equation (1) with alternating sign, i.e. positive and negative values. If M is connected, alternating sign implies that the zero set $u^{-1}(0)$ of u is not empty. In the following we will use the standard definition to call the zero set $u^{-1}(0)$ of a function u the *nodal set of u*. A solution with a non-empty nodal set is usually called a *nodal solution*. If M is connected, then the maximum principle implies that a solution of the Yamabe equation is nodal if and only if it has alternating sign. They are called *nodal solutions* of the Yamabe equation. The articles [HV94], [DJ02], [Jou99], [Hol99] prove existence of nodal solutions under symmetry assumptions or under some assumptions which allow to use Aubin's methods, as in Theorem 1.4 when $\mu_1(M,g) = 0$ and $\mu_2(M,g) < \mu_1(\mathbb{S}^n)$. If $\mu(M,g) \leq 0$, another method is given in Section 8. The method we use here is completely different and we obtain solutions on a large class of manifolds. In particular, to our knowledge, there is no work which leads to the existence of such solutions if the Yamabe invariant is positive and if (M,g) is not conformally equivalent to the round sphere. The result we obtain is the following:

Theorem 1.6. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Assume that $\mu_2(M,g)$ is attained by a generalized metric $u^{N-2}g$ where $u \in L^N(M)$, $u \geq 0$ and $u \not\equiv 0$. Let Ω be the nodal set of u. Then, there exists a nodal solution $w \in C^\infty(M \setminus \Omega) \cap C^{3,\alpha}(M)$ ($\alpha \leq N-2$) of equation (1) such that $|w| = u$.*

A corollary of Theorems 1.4, 1.5 and 1.6 is then

Corollary 1.7. *Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$ whose Yamabe invariant is non-negative. We assume that one of the following assumptions is true:*

- *$\mu_1(M,g) > 0$, (M,g) is not locally conformally flat and $n \geq 11$;*
- *$\mu_1(M,g) = 0$, (M,g) is not locally conformally flat and $n \geq 9$.*

Then, there exists a nodal solution of Yamabe equation (1).

Acknowledgement
The authors want to thank M. Ould Ahmedou for many interesting conversations about nodal solutions of the Yamabe equation. His large knowledge about such problems was a stimulating inspiration for this article. The author are also extremely obliged to Frédéric Robert for having pointed out a little mistake in the first version of this paper.

2. Variational characterization of $\mu_2(M,g)$

2.1. **Notation.** In the whole article we will use the following notations

$$L^N_+(M) := \left\{ u \in L^N(M) \,|\, u \geq 0, \quad u \not\equiv 0 \right\}.$$

2.2. **Grassmannians and the min-max principle.** Let $\mathrm{Gr}_k(C^\infty(M))$ be the k-dimensional *Grassmannian* in $C^\infty(M)$, i.e. the set of all k-dimensional subspaces of $C^\infty(M)$. The Grassmannian is an important ingredient in the min-max characterization of $\lambda_k(g)$

$$\lambda_k(L_{\tilde{g}}) := \inf_{V \in \mathrm{Gr}_k(C^\infty(M))} \sup_{v \in V \setminus \{0\}} \frac{\int (L_{\tilde{g}} v) v \, dv_{\tilde{g}}}{\int_M v^2 \, dv_{\tilde{g}}}.$$

We will also need a slightly modified Grassmannian. For any $u \in L^N_+(M)$ we define $\mathrm{Gr}^u_k(C^\infty)$ to be the set of all k-dimensional subspaces of $C^\infty(M)$, such that the restriction operator to $M \setminus u^{-1}(0)$ is injective. More explicitly, we have $\mathrm{span}(v_1, \ldots, v_k) \in \mathrm{Gr}^u_k(C^\infty(M))$ if and only if $v_1|_{M \setminus u^{-1}(0)}, \ldots, v_k|_{M \setminus u^{-1}(0)}$ are linearly independent. Sometimes it will be convenient to use the equivalent statement that the functions $u^{\frac{N-2}{2}} v_1, \ldots, u^{\frac{N-2}{2}} v_k$ are linearly independent.

Similarly, by replacing $C^\infty(M)$ by $H^2_1(M)$ we obtain the definitions of $\mathrm{Gr}_k(H^2_1(M))$ and $\mathrm{Gr}^u_k(H^2_1(M))$.

2.3. **The functionals.** For all $u \in L^N_+(M)$, $v \in H^2_1(M)$ such that $u^{\frac{N-2}{2}} v \not\equiv 0$, we set

$$F(u,v) = \frac{\int_M c_n |\nabla v|^2 + S_g v^2 \, dv_g}{\int_M v^2 u^{N-2} \, dv_g} \left(\int_M u^N \, dv_g \right)^{\frac{2}{n}}.$$

2.4. **Variational characterization of $\mu_2(M,g)$.** The following characterization will be of central importance for our article.

Proposition 2.1. *We have*

$$\mu_k(M,g) = \inf_{\substack{u \in L^N_+(M) \\ V \in \mathrm{Gr}^u_k(H^2_1(M))}} \sup_{v \in V \setminus \{0\}} F(u,v) \tag{2}$$

Proof. Let u be a smooth positive function on M. For all smooth functions f, $f \not\equiv 0$, we set $\tilde{g} = u^{N-2} g$ ($N = \frac{2n}{n-2}$) and

$$F'(u,f) = \frac{\int_M f L_{\tilde{g}} f \, dv_{\tilde{g}}}{\int_M f^2 \, dv_{\tilde{g}}}.$$

The operator L_g is conformally invariant (see [Heb97]) in the following sense:

$$u^{N-1} L_{\tilde{g}}(u^{-1} f) = L_g(f) \tag{3}$$

Together with the fact that

$$dv_{\tilde g} = u^N \, dv_g, \tag{4}$$

we get that

$$F'(u,f) = \frac{\int_M (uf) L_g (uf) \, dv_g}{\int_M (uf)^2 u^{N-2} \, dv_g}.$$

Using the min-max principle, we can write that

$$\lambda_k(\tilde g) = \inf_{V \in \mathrm{Gr}_k^u(H_1^2(M))} \sup_{f \in V \setminus \{0\}} F'(u,f)$$

Now, replacing uf by v, we obtain that

$$\lambda_k(\tilde g) = \inf_{V \in \mathrm{Gr}_k(H_1^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v L_g v \, dv_g}{\int_M v^2 u^{N-2} \, dv_g}. \tag{5}$$

Using the definition of μ_2 and $\mathrm{Vol}_{\tilde g}(M) = \int_M u^N \, dv_g$, we derive

$$\mu_k(M,g) = \inf_{\substack{u \in L_+^N(M) \\ V \in \mathrm{Gr}_k^u(C^\infty(M))}} \sup_{v \in V \setminus \{0\}} F(u,v)$$

The result follows immediately.

3. Generalized metrics and the Euler-Lagrange equation

3.1. **A regularity result.** We will need the following result.

Lemma 3.1. *Let $u \in L^N(M)$ and $v \in H_1^2(M)$. We assume that*

$$L_g v = u^{N-2} v$$

holds in the sense of distributions. Then, $v \in L^{N+\varepsilon}(M)$ for some $\varepsilon > 0$.

This result is well known for the standard Yamabe equation. Proofs for the standard Yamabe equation can be found in [Tru68] and [Heb97], and the modifications for proving Lemma 3.1 are obvious. Unfortunately, [Tru68] contains some typos, and the book [Heb97] is difficult to obtain. This is why we included a proof in the appendix for the convenience of the reader.

3.2. **The k-th eigenvalue of the Yamabe operator for a generalized metric.** On a given Riemannian manifold (M,g) we say that $\tilde g = u^{N-2} g$, $u \in L_+^N(M)$, is a *generalized metric* conformal to g. For a generalized metric $\tilde g$, we can define

$$\lambda_k(\tilde g) = \inf_{V \in \mathrm{Gr}_k^u(H_1^2(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v L_g v \, dv_g}{\int_M v^2 u^{N-2} \, dv_g}. \tag{6}$$

Proposition 3.2. *For any $u \in L_+^N$, $\tilde g = u^{N-2}$ there exist two functions v, w belonging to $H_1^2(M)$ with $v \geq 0$ and such that in the sense of distributions.*

$$L_g v = \lambda_1(\tilde g) u^{N-2} v \tag{7}$$

and

$$L_g w = \lambda_2(\tilde g) u^{N-2} w. \tag{8}$$

Moreover, we can normalize v, w by

$$\int_M u^{N-2} v^2 \, dv_g = \int_M u^{N-2} w^2 \, dv_g = 1 \text{ and } \int_M u^{N-2} v w \, dv_g = 0 \tag{9}$$

For $k = 2$ the infimum in formula (5) over all subspaces $V \in \mathrm{Gr}_2^u(H_1^2(M))$ is attained by $V = \mathrm{span}(v, w)$ and the supremum over the functions in $V \setminus \{0\}$ is attained by w. The reader should pay attention to the fact that the space V is in general non unique. As one can check, if w changes the sign then the supremum over all $v \in V = \mathrm{span}(v, w) \setminus \{0\}$ and the supremum over all $v \in V_1 = \mathrm{span}(w, |w|) \setminus \{0\}$ coincide.

From section (2), we get

$$\mu_2(M, g) = \inf_{\tilde{g} \in \overline{[g]}} \lambda_2(\tilde{g}).$$

Hence, $\mu_2(M, g)$ can be attained by a regular metric, or by a generalized metric or it can be not attained at all. These questions are discussed in Section 5. Let us now prove Proposition 3.2.

Proof of Proposition 3.2: Let $(v_m)_m$ be a minimizing sequence for $\lambda_1(\tilde{g})$ i.e. a sequence $v_m \in H_1^2(M)$ such that

$$\lim_{m \to \infty} \frac{\int_M c_n |\nabla v_m|^2 + S_g v_m^2 \, dv_g}{\int_M u^{N-2} v_m^2 \, dv_g} = \lambda_1(\tilde{g}).$$

It is well known that $(|v_m|)_m$ is also a minimizing sequence. Hence, we can assume that $v_m \geq 0$. If we normalize v_m by $\int_M u^{N-2} v_m^2 \, dv_g = 1$, then $(v_m)_m$ is bounded in $H_1^2(M)$ and after restriction to a subsequence we may assume that there exists $v \in H_1^2(M)$, $v \geq 0$ such that $v_m \to v$ weakly in $H_1^2(M)$, strongly in $L^2(M)$ and almost everywhere. If u is smooth, then

$$\int_M u^{N-2} v^2 \, dv_g = \lim_m \int_M u^{N-2} v_m^2 \, dv_g = 1 \tag{10}$$

and by standard arguments, v is a non-negative minimizer of the functional associated to $\lambda_1(\tilde{g})$. We must show that (10) still holds if $u \in L_+^N(M)$. Let $A > 0$ be a large real number and set $u_A = \inf(u, A)$. Then, using the Hölder inequality, we write

$$\begin{aligned}
\left| \int_M u^{N-2} \left(v_m^2 - v^2 \right) \, dv_g \right| &\leq \left(\int_M u_A^{N-2} |v_m^2 - v^2| \, dv_g + \int_M (u^{N-2} - u_A^{N-2})(|v_m| + |v|)^2 \, dv_g \right) \\
&\leq A \int_M |v_m^2 - v^2| \, dv_g \\
&\quad + \left(\int_M (u^{N-2} - u_A^{N-2})^{\frac{N}{N-2}} \, dv_g \right)^{\frac{N-2}{N}} \left(\int_M (|v_m| + |v|)^N \, dv_g \right)^{\frac{2}{N}}.
\end{aligned}$$

By Lebesgue's theorem we see that

$$\lim_{A \to +\infty} \int_M (u^{N-2} - u_A^{N-2})^{\frac{N}{N-2}} \, dv_g = 0.$$

Since $(v_m)_m$ is bounded in $H_1^2(M)$, it is bounded in $L^N(M)$ and hence there exists $C > 0$ such that $\int_M (|v_m| + |v|)^N \, dv_g \leq C$. By strong convergence in $L^2(M)$,

$$\lim_m \int_M |v_m^2 - v^2| \, dv_g = 0.$$

Equation (10) easily follows and v is a non-negative minimizer of the functional associated to $\lambda_1(\tilde{g})$. Writing the Euler-Lagrange equation of v, we find that v satisfies equation (7). Now, we define

$$\lambda_2'(\tilde{g}) = \inf \frac{\int_M c_n |\nabla w|^2 + S_g w^2 \, dv_g}{\int_M u^{N-2} |w|^2 \, dv_g}$$

where the infimum is taken over smooth functions w such that $u^{\frac{N-2}{2}} w \not\equiv 0$ and such that $\int_M u^{N-2} v w \, dv_g = 0$. With the same method, we find a minimizer w of this problem that satisfies (8) with $\lambda_2'(\tilde{g})$ instead of $\lambda_2(\tilde{g})$. However, it is not difficult to see that $\lambda_2'(\tilde{g}) = \lambda_2(\tilde{g})$ and Proposition 3.2 easily follows.

3.3. Euler-Lagrange equation of a minimizer of $\lambda_2 \mathrm{Vol}^{2/n}$.

Lemma 3.3. *Let $u \in L^N_+(M)$ with $\int u^N = 1$. Suppose that $w_1, w_2 \in H_1^2(M) \setminus \{0\}$, $w_1, w_2 \geq 0$ satisfy*

$$\int (c_n|\nabla w_1|^2 + \mathrm{Scal}_g w_1^2)\, dv_g \leq \mu_2(M,g) \int u^{N-2} w_1^2 \tag{11}$$

$$\int (c_n|\nabla w_2|^2 + \mathrm{Scal}_g w_2^2)\, dv_g \leq \mu_2(M,g) \int u^{N-2} w_2^2 \tag{12}$$

and suppose that $(M \setminus w_1^{-1}(0)) \cap (M \setminus w_2^{-1}(0))$ has measure zero. Then u is a linear combination of w_1 and w_2 and we have equality in (11) *and* (12).

Proof. We let $\bar{u} = aw_1 + bw_2$ where $a, b > 0$ are chosen such that

$$\frac{a^{N-2}}{b^{N-2}} \frac{\int_M u^{N-2} w_1^2\, dv_g}{\int_M u^{N-2} w_2^2\, dv_g} = \frac{\int_M w_1^N\, dv_g}{\int_M w_2^N\, dv_g} \tag{13}$$

and

$$\int_M \bar{u}^N\, dv_g = a^N \int_M w_1^N + b^N \int w_2^N = 1. \tag{14}$$

Because of the variational characterization of μ_2 we have

$$\mu_2(M,g) \leq \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(\bar{u}, \lambda w_1 + \mu w_2) \tag{15}$$

By (11),(12) and (14), and since $(M \setminus w_1^{-1}(0)) \cap (M \setminus w_2^{-1}(0))$ has measure zero

$$\begin{aligned} F(\bar{u}, \lambda w_1 + \mu w_2) &= \frac{\lambda^2 \int_M \left(c_n|\nabla w_1|^2 + S_g w_1^2\right) dv_g + \mu^2 \int_M \left(c_n|\nabla w_2|^2 + S_g w_2^2\right) dv_g}{\lambda^2 \int_M |\bar{u}|^{N-2} w_1^2\, dv_g + \mu^2 \int_M |\bar{u}|^{N-2} w_2^2\, dv_g} \\ &\leq \mu_2(M,g) \frac{\lambda^2 \int_M u^{N-2} w_1^2\, dv_g + \mu^2 \int_M u^{N-2} w_2^2\, dv_g}{\lambda^2 a^{N-2} \int_M w_1^N\, dv_g + \mu^2 b^{N-2} \int_M w_2^N\, dv_g}. \end{aligned} \tag{16}$$

As one can check, relation (13) implies that this expression does not depend on λ, μ. Hence, setting $\lambda = a$ and $\mu = b$, the denominator is 1, and we get

$$\begin{aligned} \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(\bar{u}, \lambda w_1 + \mu w_2) &\leq \mu_2(M,g) \int_M u^{N-2}(a^2 w_1^2 + b^2 w_2^2)\, dv_g \\ &= \mu_2(M,g) \int_M u^{N-2} \bar{u}^2\, dv_g. \end{aligned}$$

By Hölder inequality,

$$\sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(\bar{u}, \lambda w_1 + \mu w_2) \leq \mu_2(M,g) \left(\int_M u^N\, dv_g\right)^{\frac{N-2}{N}} \left(\int_M \bar{u}^N\, dv_g\right)^{\frac{2}{N}} = \mu_2(M,g). \tag{17}$$

Inequality (15) implies that we have both equality in the Hölder inequality of (17) and in (16). The equality in the Hölder inequality implies that there exists a constant $c > 0$ such that $u = c\bar{u}$ almost everywhere. Moreover, since $\int u^N = \int \bar{u}^N = 1$, we have $u = \bar{u} = aw_1 + bw_2$. The equality in (16) implies inequality in (11) and (12). ☐

Theorem 3.4 (Euler-Lagrange equation)**.** *Assume that $\mu_2(M,g) \neq 0$ and that $\mu_2(M,g)$ is attained by a generalized metric $\tilde{g} = u^{N-2}g$ with $u \in L^N_+(M)$. Let v, w be as in Proposition 3.2. Then, $u = |w|$. In particular,*

$$L_g w = \mu_2(M,g)|w|^{N-2} w \tag{18}$$

Moreover, w has alternating sign and $w \in C^{3,\alpha}(M)$ ($\alpha \leq N-2$).

Remark 3.5. Assume that $\mu_2(M,g)$ is equal to 0 and is attained by a generalized metric g', then, using the conformal invariance of the Yamabe operator, it is easy to check that for all generalized metrics $\tilde{g}$ conformal to g', we have $\lambda_2(\tilde{g}) = 0$. Consequently, each metric conformal to g is a minimizer for $\mu_2(M,g)$ and Theorem 3.4 is always false in this case. However, we will still get a nodal solution of (1) if $\mu_2(M,g) = 0$. Indeed, by Theorem 1.4 and the remark above, $\lambda_2(g) = 0$. Let w be an eigenfunction associated to $\lambda_2(g)$. We have $L_g w = 0$. Then, we have a solution of (18).

Remark 3.6. Assume that $\mu_2(M,g) \neq 0$ and that $\mu_2(M,g)$ is attained by a generalized metric. Let w be the solution of equation (18) given by Theorem 3.4. We let $\Omega_+ = \{x \in M \text{ s.t. } w(x) > 0\}$ and $\Omega_- = \{x \in M \text{ s.t. } w(x) - 0\}$. Then, a immediate consequence of Lemma 3.3 is that Ω_+ and Ω_- have exactly one connex component.

Proof. Without loss of generality, we can assume that $\int_M u^N\, dv_g = 1$. By assumption we have $\lambda_2(\tilde{g}) = \mu_2(M,g)$. Let $v, w \in H_1^2(M)$ be some functions satisfying equations (7), (8) and relation (9).

Step 1. *We have $\lambda_1(\tilde{g}) < \lambda_2(\tilde{g})$.*

We assume that $\lambda_1(\tilde{g}) = \lambda_2(\tilde{g})$. Then, after possibly replacing w by a linear combination of v and w, we can assume that the function $u^{\frac{N-2}{2}} w$ changes the sign. We apply Lemma 3.3 for $w_1 := \sup(w,0)$ and $w_2 := \sup(-w,0)$. We obtain the existence of $a, b > 0$ with $u = aw_1 + bw_2$. Now, by Lemma 3.1, $w \in L^{N+\varepsilon}(M)$. By a standard bootstrap argument, equation (8) shows that $w \in C^{2,\alpha}(M)$ for all $\alpha \in]0,1[$. It follows that $u \in C^{0,\alpha}(M)$ for all $\alpha \in]0,1[$. Now, since $\lambda_1(\tilde{g}) = \lambda_2(\tilde{g})$ and by definition of $\lambda_1(\tilde{g})$, w is a minimizer of the functional $\bar{w} \mapsto F(u, \bar{w})$ among the functions belonging to $H_1^2(M)$ and such that $u^{\frac{N-2}{2}} \bar{w} \not\equiv 0$. Since $F(u,w) = F(u,|w|)$, we see that $|w|$ is a minimizer for the functional associated to $\lambda_1(\tilde{g})$ and hence, writing the Euler-Lagrange equation of the problem, w satisfies the same equation as w. As a consequence, $|w|$ is $C^2(M)$. By the maximum principle, we get $|w| > 0$ everywhere. This is false. Hence, the step is proved.

Step 2. *The function w changes the sign.*

Assume that w does not change the sign, i.e. after possibly replacing w by $-w$, we have $w \geq 0$. Using (9) we see that $(M \setminus v^{-1}(0)) \cap (M \setminus w^{-1}(0))$ has measure zero. Setting $w_1 := v$ and $w_2 := w$ we have (11) and (12). While we have equality in (12), Step 1 implies that inequality (11) is strict. However using Lemma 3.3 we can derive equality in (11). Hence we obtain a contradiction, and the step is proved.

Step 3. *There exists $a, b > 0$ such that $u = a\sup(w,0) + b\sup(-w,0)$. Moreover, $w \in C^{2,\alpha}(M)$ and $u \in C^{0,\alpha}(M)$ for all $\alpha \in]0,1[$.*

As in the proof of Step 1 we apply Lemma 3.3 for $w_1 := \sup(w,0)$ and $w_2 := \sup(-w,0)$. We obtain the existence of $a, b > 0$ such that $u = aw_1 + bw_2$. As in Step 1 we get that $w \in C^{2,\alpha}(M)$ and $u \in C^{0,\alpha}(M)$ for all $\alpha \in]0,1[$. This proves the present step.

Step 4. *Conclusion.*

Let $h \in C^\infty(M)$ whose support is contained in $M \setminus \{u^{-1}(0)\}$. For t close to 0, set $u_t = |u + th|$. Since $u > 0$ on the support of h and since u is continuous (see last step), we have for t close to 0, $u_t = u + th$. As $\mathrm{span}(v,w) \in \mathrm{Gr}_2^u(H_1^2(M))$ we obtain using (2) for all t

$$\mu_2(M,g) \leq \sup_{(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\}} F(u_t, \lambda v + \mu w).$$

Equations (7), (8), and relation (9) yield

$$\begin{aligned} F(u_t, \lambda v + \mu w) &= \frac{\lambda^2 \lambda_1(\tilde{g}) \int_M u^{N-2} v^2\, dv_g + \mu^2 \lambda_2(\tilde{g}) \int_M u^{N-2} w^2\, dv_g}{\lambda^2 \int_M u_t^{N-2} v^2\, dv_g + 2\lambda\mu \int_M u_t^{N-2} vw\, dv_g + \mu^2 \int_M u_t^{N-2} w^2\, dv_g} \left(\int_M u_t^N\, dv_g \right)^{\frac{2}{n}} \\ &= \frac{\lambda^2 \lambda_1(\tilde{g}) + \mu^2 \lambda_2(\tilde{g})}{\lambda^2 a_t + \lambda\mu b_t + \mu^2 c_t} \left(\int_M |u_t|^N\, dv_g \right)^{\frac{2}{n}}. \end{aligned}$$

where

$$a_t = \int_M u_t^{N-2} v^2 \, dv_g,$$
$$b_t = 2 \int u_t^{N-2} vw \, dv_g$$

and

$$c_t = \int_M u_t^{N-2} w^2 \, dv_g.$$

The functions a_t, b_t and c_t are smooth for t close to 0, furthermore $a_0 = c_0 = 1$ and $b_0 = 0$. The function $f(t,\alpha) := F(u_t, \sin(\alpha)v + \cos(\alpha)w)$ is smooth for small t. Using $\lambda_1(\tilde g) < \lambda_2(\tilde g)$ one calculates

$$\begin{array}{lll} \dfrac{\partial}{\partial\alpha} f(0,\alpha) = 0 & \Leftrightarrow & \alpha \in \dfrac{\pi}{2}\mathbb{Z} \\ \dfrac{\partial^2}{\partial\alpha^2} f(0,\alpha) < 0 & \text{for} & \alpha \in \pi\mathbb{Z} \\ \dfrac{\partial^2}{\partial\alpha^2} f(0,\alpha) > 0 & \text{for} & \alpha \in \pi\mathbb{Z} + \dfrac{\pi}{2} \end{array}$$

Applying the implicit function theorem to $\frac{\partial f}{\partial \alpha}$ at the point $(0,0)$, we see that there is a smooth function $t \mapsto \alpha(t)$, defined on a neighborhood of 0 with $\alpha(0) = 0$ and

$$f(t,\alpha(t)) = \sup_{\alpha\in\mathbb{R}} f(t,\alpha) = \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(u_t, \lambda v + \mu w).$$

As a consequence

$$\frac{d}{dt}|_{t=0} \sin^2\alpha(t) = \frac{d}{dt}|_{t=0} \cos^2\alpha(t) = \frac{d}{dt}|_{t=0}(\sin^2\alpha(t) a_t) = \frac{d}{dt}|_{t=0}(\sin\alpha(t)\cos\alpha(t) b_t) = 0.$$

Hence, $\frac{d}{dt}|_{t=0} f(t,\alpha(t))$ exists and we have

$$\begin{aligned} \frac{d}{dt}|_{t=0} f(t,\alpha(t)) &= \lambda_2(M,\tilde g)\left(-\frac{d}{dt}|_{t=0} c_t + \frac{d}{dt}|_{t=0}\left(\int_M |u_t|^N \, dv_g\right)^{\frac{2}{n}}\right) \\ &= \lambda_2(M,\tilde g)(N-2)\left(-\int_M u^{N-3} h w^2 \, dv_g + \int_M u^{N-1} h \, dv_g\right). \end{aligned}$$

By definition of $\mu_2(M,g)$, f admits a minimum in $t = 0$. As $\lambda_2(M,\tilde g) = \mu_2(M,g) \neq 0$ we obtain

$$\int_M u^{N-3} h w^2 \, dv_g = \int_M u^{N-1} h \, dv_g.$$

Since h is arbitrary (we just have to ensure that its support is contained in $M \setminus \{u^{-1}(0)\}$), we get that $u^{N-3}w^2 = u^{N-1}$ on $M \setminus \{u^{-1}(0)\}$, hence $u = |w|$ on $M \setminus \{u^{-1}(0)\}$. Together with Step 3, we get $u = |w|$ everywhere. This proves theorem 3.4. □

4. A sharp Sobolev inequality related to $\mu_2(M,g)$

4.1. **Statement of the results.** For any compact Riemannian manifold (M,g) of dimension $n \geq 3$, Hebey and Vaugon have shown in ([HV96]) that there exists $B_0(M,g) > 0$ such that

$$\mu_1(\mathbb{S}^n) = n(n-1)\,\omega_n^{\frac{2}{n}} = \inf_{u\in H_1^2(M)\setminus\{0\}} \frac{\int_M c_n|\nabla u|^2 + B_0\int_M u^2 \, dv_g}{\left(\int_M u^N \, dv_g\right)^{\frac{2}{n}}} \tag{S}$$

where ω_n stands for the volume of the standard n-dimensional sphere $\mathbb{S}^n$ and where $\mu_1(\mathbb{S}^n)$ is the Yamabe invariant of $\mathbb{S}^n$.

This inequality is strongly related to the resolution of the Yamabe problem. It allows to avoid concentration for the minimizing sequence of $\mu_1(M,g)$. For the minimization of $\mu_2(M,g)$, this inequality is not sufficient and another one must be constructed. The following result is adapted to the problem of minimizing $\mu_2(M,g)$.

Theorem 4.1. *On a compact connected Riemannian manifold (M,g) of dimension $n \geq 3$ we have*

$$2^{\frac{2}{n}}\mu_1(\mathbb{S}^n) = \inf_{\substack{u\in L_+^N(M)\\ V\in \mathrm{Gr}_k^u(H_1^2(M))}} \sup_{v\in V\setminus\{0\}} \frac{\left(\int_M c_n|\nabla v|^2\,dv_g + B_0(M,g)\int_M v^2\,dv_g\right)\left(\int_M u^N\,dv_g\right)^{\frac{2}{N}}}{\int_M u^{N-2}v^2\,dv_g} \tag{S$_1$}$$

where $B_0(M,g)$ is given by inequality (S).

We present now two corollaries of Theorem 4.1.

Corollary 4.2. *For the standard n-dimensional sphere we have $\mu_2(\mathbb{S}^n) = 2^{\frac{2}{n}}\mu_1(\mathbb{S}^n)$.*

Corollary 4.3. *For all $u \in C_c^\infty(\mathbb{R}^n)$ and $V \in \mathrm{Gr}_2^u(C_c^\infty(\mathbb{R}^n))$ we have*

$$2^{2/n}\mu_1(\mathbb{S}^n) \leq \sup_{v\in V\setminus\{0\}} \frac{\left(\int_{\mathbb{R}^n} c_n|\nabla v|^2\,dv_g\right)\left(\int_{\mathbb{R}^n}|u|^N\,dv_g\right)^{\frac{2}{N}}}{\int_{\mathbb{R}^n}|u|^{N-2}v^2\,dv_g}$$

4.2. **Proof of theorem 4.1.** The functional

$$G(u,v) := \frac{\left(\int_M c_n|\nabla v|^2\,dv_g + B_0(M,g)\int_M v^2\,dv_g\right)\left(\int_M u^N\,dv_g\right)^{\frac{2}{N}}}{\int_M u^{N-2}v^2\,dv_g}$$

is continuous on $L_+^N(M) \times (H_1^2(M)\setminus\{0\})$. As a consequence $I(u,V) := \sup_{v\in V\setminus\{0\}} G(u,v)$ depends continuously on $u \in L_+^N(M)$ and $V \in \mathrm{Gr}_2^u(H_1^2(M))$. Thus, in order to show the theorem it is sufficient to show that $I(u,V) \geq 2^{2/n}\mu_1(\mathbb{S}^n)$ for all smooth $u>0$ and $V \in \mathrm{Gr}_2(C^\infty(M))$. Without loss of generality, we can assume

$$\int_M u^N\,dv_g = 1. \tag{19}$$

The operator $v \mapsto P(v) := c_n u^{\frac{2-N}{2}}\Delta(u^{\frac{2-N}{2}}v) + B_0(M,g)u^{2-N}v$ is an elliptic operator on M, and P is self-adjoint with respect to the L^2-scalar product. Hence, P has discrete spectrum $\lambda_1 \leq \lambda_2 \leq \dots$ and the corresponding eigenfunctions $\varphi_1, \varphi_2, \dots$ are smooth. Setting $v_i := u^{\frac{2-N}{2}}\varphi_i$ we obtain

$$(c_n\Delta + B_0)(v_i) = \lambda_i u^{N-2}v_i$$

$$\int u^{N-2}v_iv_j\,dv_g = 0 \qquad \text{if } \lambda_i \neq \lambda_j.$$

The maximum principle implies that an eigenfunction to the smallest eigenvalue λ_1 has no zeroes. Hence $\lambda_1 < \lambda_2$, and we can assume $v_1 > 0$.

We define $w_+ := a_+\sup(0,v_2)$ and $w_- := a_-\sup(0,-v_2)$, where we choose $a_+, a_- > 0$ such that

$$\int_M u^{N-2}w_-^2\,dv_g = \int_M u^{N-2}w_+^2\,dv_g = 1.$$

We let $\Omega_- = \{w<0\}$ and $\Omega_+ = \{w \geq 0\}$. By Hölder inequality,

$$\begin{aligned} 2 &= \int_M u^{N-2}w_-^2\,dv_g + \int_M u^{N-2}w_+^2\,dv_g \\ &\leq \left(\int_{\Omega_-} u^N\,dv_g\right)^{\frac{N-2}{N}}\left(\int_M w_-^N\,dv_g\right)^{\frac{2}{N}} + \left(\int_{\Omega_+} u^N\,dv_g\right)^{\frac{N-2}{N}}\left(\int_M w_+^N\,dv_g\right)^{\frac{2}{N}}. \end{aligned}$$

Using the sharp Sobolev inequality (S), we get that

$$2\mu_1(\mathbb{S}^n) \leq \left(\int_{\Omega_-} u^N\,dv_g\right)^{\frac{N-2}{N}} \int_M w_- u^{\frac{N-2}{2}} P\left(u^{\frac{N-2}{2}}w_-\right)\,dv_g \tag{20}$$

$$+ \left(\int_{\Omega_+} u^N\,dv_g\right)^{\frac{N-2}{N}} \int_M w_+ u^{\frac{N-2}{2}} P\left(u^{\frac{N-2}{2}}w_+\right)\,dv_g \tag{21}$$

Since w_- resp. w_+ are some multiples of w on Ω_- resp. Ω_+, they satisfy the same equation as w. Hence, we get that

$$\begin{aligned} 2 &= \mu_1(\mathbb{S}^n)^{-1}\lambda_2\left(\left(\int_{\Omega_-} u^N\,dv_g\right)^{\frac{N-2}{N}}\int_M u^{N-2}w_-^2\,dv_g + \left(\int_{\Omega_+} u^N\,dv_g\right)^{\frac{N-2}{N}}\int_M u^{N-2}w_+^2\,dv_g\right) \\ &= \mu_1(\mathbb{S}^n)^{-1}\lambda_2\left(\left(\int_{\Omega_-} u^N\,dv_g\right)^{\frac{N-2}{N}} + \left(\int_{\Omega_+} u^N\,dv_g\right)^{\frac{N-2}{N}}\right). \end{aligned}$$

Now, for any real non-negative numbers $a, b \geq 0$, the Hölder inequality yields

$$a + b \leq 2^{\frac{2}{N}}\left(a^{\frac{N}{N-2}} + b^{\frac{N}{N-2}}\right)^{\frac{N-2}{N}}$$

We apply this inequality with $a = \left(\int_{\Omega_-} u^N\,dv_g\right)^{\frac{N-2}{N}}$ and $b = \left(\int_{\Omega_+} u^N\,dv_g\right)^{\frac{N-2}{N}}$. Using (19), we obtain

$$2 \leq 2^{\frac{2}{N}}\mu_1(\mathbb{S}^n)^{-1}\lambda_2\left(\int_{\Omega_-} u^N\,dv_g + \int_{\Omega_+} u^N\,dv_g\right) = 2^{\frac{2}{N}}\mu_1(\mathbb{S}^n)^{-1}\lambda_2.$$

We obtain $\lambda_2 \geq 2^{\frac{2}{n}}\mu(\mathbb{S}^n)$. Since $\lambda_2 = I(u, \mathrm{span}(v_1, v_2))$, this ends the proof of Theorem 4.1.

4.3. **Proof of Corollaries 4.2 and 4.3.** It is well known that $B_0(\mathbb{S}^n)$ equals to the scalar curvature of $\mathbb{S}^n$, i.e. $B_0(\mathbb{S}^n) = n(n-1)$. Replacing $B_0(\mathbb{S}^n)$ by its value and taking the infimum over u, V, the right hand term of inequality (S_1) is exactly the variational characterization of $\mu_2(\mathbb{S}^n)$ (see equation (2)). This proves that $\mu_2(\mathbb{S}^n) \geq 2^{2/n}\mu_1(\mathbb{S}^n)$. Corollary 4.2 then follows from Theorem 5.4. Since $\mathbb{R}^n$ is conformal to $\mathbb{S}^n \setminus \{p\}$ (p is any point of $\mathbb{S}^n$), we can use the conformal invariance to prove Corollary 4.3.

5. Some properties of $\mu_2(M, g)$

5.1. **Is $\mu_2(M, g)$ attained?** Let (M, g) be an n-dimensional compact Riemannian manifold. The Yamabe problem shows that $\mu_1(M, g)$ is attained by a metric $\tilde{g}$ conformal to g. Some questions arise naturally concerning $\mu_2(M, g)$:

1- Is $\mu_2(M, g)$ attained by a metric?

2- Is it possible that $\mu_2(M, g)$ is attained by a generalized metric?

In this section, we give answers to these questions. The first result we prove is the following:

Proposition 5.1. *Let $\mathbb{S}^n \dot{\cup} \mathbb{S}^n$ be the disjoint union of two copies of the sphere equipped with their standard metric. Then, $\mu_2(\mathbb{S}^n \dot{\cup} \mathbb{S}^n) = 2^{2/n}\mu_1(\mathbb{S}^n)$ and it is attained by the canonical metric.*

Proof. One computes

$$\lambda_2(\mathbb{S}^n \dot{\cup} \mathbb{S}^n)\,\mathrm{Vol}(\mathbb{S}^n \dot{\cup} \mathbb{S}^n)^{2/n} = 2^{2/n}\lambda_1(\mathbb{S}^n)\,\mathrm{Vol}(\mathbb{S}^n)^{2/n} = 2^{2/n}\mu_1(\mathbb{S}^n).$$

Hence $\mu_2(\mathbb{S}^n \dot{\cup} \mathbb{S}^n) \leq 2^{2/n}\mu_1(\mathbb{S}^n)$ follows.

Now, let $\tilde{g}$ be an arbitrary smooth metric on $S^n \dot{\cup} S^n$. We write S_1^n for the first S^n and S_2^n for the second S^n. Then $\lambda_2(S^n \dot{\cup} S^n, \tilde{g})$ is the minimum of $\lambda_2(S_1^n, \tilde{g})$, $\lambda_2(S_2^n, \tilde{g})$ and $\max\{\lambda_1(S_1^n, \tilde{g}), \lambda_1(S_2^n, \tilde{g})\}$.

It follows from Corollary 4.2 that

$$\lambda_2(S_1^n, \tilde{g})\,\mathrm{Vol}(S^n \dot{\cup} S^n, \tilde{g})^{2/n} \geq \lambda_2(S_1^n, \tilde{g})\,\mathrm{Vol}(S_1^n, \tilde{g})^{2/n} \geq 2^{2/n}\mu_1(\mathbb{S}^n),$$

and obviously we have the same for $\lambda_2(S_2^n, \tilde{g})$.

Summing

$$\lambda_1(S_i^n, \tilde{g})^{n/2} \geq \mu_1(\mathbb{S}^n)^{n/2}\mathrm{Vol}(S_i^n, \tilde{g})$$

over $i \in \{1, 2\}$, we obtain the remaining inequality

$$\max\{\lambda_1(S_1^n, \tilde{g}), \lambda_1(S_2^n, \tilde{g})\}\,\mathrm{Vol}(S^n \dot{\cup} S^n, \tilde{g})^{2/n} \geq 2^{2/n}\mu_1(\mathbb{S}^n),$$

and the proposition is proved. ∎

Question 1 is solved by the following result.

Proposition 5.2. *If M is connected, then $\mu_2(M,g)$ cannot be attained by a metric.*

Indeed, otherwise by Theorem 3.4, we would have that $u = |w|$ and hence u cannot be positive. Theorem 1.4 and the following result answer Question 2.

Proposition 5.3. *The invariant $\mu_2(\mathbb{S}^n)$ is not attained by a generalized metric.*

This proposition immediately follows from Proposition 5.6.

5.2. **Some bounds of $\mu_2(M,g)$.** At first, we give an upper bound for $\mu_2(M,g)$.

Theorem 5.4. *Let (M,g) be an n-dimensional compact Riemannian manifold with $\mu_1(M,g) \geq 00$. Then,*

$$\mu_2(M,g) \leq \left(\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}}\right)^{\frac{2}{n}}. \tag{22}$$

This inequality is strict in the following cases:

- *$\mu_1(M,g) > 0$, (M,g) is not locally conformally flat and $n \geq 11$;*
- *$\mu_1(M,g) = 0$, (M,g) is not locally conformally flat and $n \geq 9$.*

From the solution of the Yamabe problem by Aubin and Schoen [Aub76, Sch84] we know that if (M,g) is not conformally equivalent to $\mathbb{S}^n$, then $\mu_1(M,g) < \mu_1(\mathbb{S}^N)$. Hence, (22) implies the following corollary.

Corollary 5.5. *Let (M,g) be an n-dimensional compact connected Riemannian manifold whose Yamabe invariant is non-negative. Then $\mu_2(M,g) \leq \mu_2(\mathbb{S}^n)$ with inequality if and only if (M,g) is conformally diffeomorphic to the sphere $\mathbb{S}^n$.*

These inequalities are very important, because they can be used to avoid concentration of minimizing sequences for μ_2, in a way which is similar to the resolution of the Yamabe problem.

The following proposition gives a lower bound for μ_2.

Proposition 5.6. *Let (M,g) be a n-dimensional compact Riemannian manifold whose Yamabe invariant is non-negative. Then,*

$$\mu_2(M,g) \geq 2^{\frac{2}{n}}\mu_1(M,g). \tag{23}$$

Moreover, if M is connected and if $\mu_2(M,g)$ is attained by a generalized metric, then this inequality is strict.

When $\mu_1(M,g) = 0$, inequality (23) is trivial. If $\mu_1(M,g) > 0$, by a possible chande of metric in the conformal class, we can assume that the scalar curvature is positive. The proof of inequality (23) is exactly the same as the one of Theorem 4.1. We just have to replace $B_0(M,g)$ by S_g. Moreover, if M were connected and if $\mu_2(M,g)$ were attained by a generalized metric, then inequality (20) would be an equality and we would have that w_+ or w_- is a function for which equality in the Sobolev inequality (S) is attained. By the maximum principle, we would get that w_+ or w_- is positive on M which is impossible.

5.2.1. *Proof of theorem 5.4.*

Lemma 5.7. *For any $\alpha > 2$, there is a $C > 0$ such that*

$$|a+b|^\alpha \leq a^\alpha + b^\alpha + C(a^{\alpha-1}b + ab^{\alpha-1})$$

for all $a, b > 0$.

Proof of Lemma 5.7. Without loss of generality, we can assume that $a = 1$. Then we set for $x > 0$,

$$f(x) = \frac{|1+x|^\alpha - (1+x^\alpha)}{x^{\alpha-1}+x}.$$

One checks that $\lim_{x\to 0} f(x) = \lim_{x\to +\infty} f(x) = \alpha$. Since f is continuous, f is bounded by a constant C on $\mathbb{R}_+$. Clearly, this constant is the desired C in inequality of Lemma 5.7.

Proof of Theorem 5.4. For $u \in H_1^2(M) \setminus \{0\}$ let

$$Y(u) = \frac{\int_M c_n |\nabla u|^2 + S_g u^2 \, dv_g}{\left(\int_M |u|^N \, dv_g\right)^{\frac{2}{N}}}$$

be the Yamabe functional of M. The solution of the Yamabe problem provides the existence of a smooth positive minimizer v of Y, and we can assume

$$\int_M v^N \, dv_g = 1. \tag{24}$$

Then, v satisfies the Yamabe equation

$$L_g v = \mu_1(M,g) v^{N-1}. \tag{25}$$

Let $x_0 \in M$ be fixed and choose a system $(x_1, \cdots, x_n)$ of normal coordinates at x_0. We note $r = dist_g(x_0, .)$. If $\delta > 0$ is a small fixed number, let η be a smooth cut-off function such that $0 \leq \eta \leq 1$, $\eta(B(x_0,\delta)) = \{1\}$ and $\eta(M \setminus B(x_0, 2\delta) = \{0\}$, $|\nabla \eta| \leq 2/\delta$. Then, we can define for all $\varepsilon > 0$

$$v_\varepsilon = C_\varepsilon \eta (\varepsilon + r^2)^{\frac{2-n}{n}}.$$

where $C_\varepsilon > 0$ is such that

$$\int_M v_\varepsilon^N \, dv_g = 1. \tag{26}$$

By standard computations (see [Aub76])

$$\lim_{\varepsilon \to 0} Y(v_\varepsilon) = \mu_1(\mathbb{S}^n). \tag{27}$$

If (M,g) is not locally conformally flat, if g is well chosen in the conformal class and if x_0 is well chosen in M, it was also proven in [Aub76] that there exists a constant $C(M) > 0$ such that

$$Y(v_\varepsilon) = \left| \begin{array}{ll} \mu_1(\mathbb{S}^n) - C(M)\varepsilon^2 + o(\varepsilon^2) & \text{if } n > 6 \\ \mu_1(\mathbb{S}^n) - C(M)\varepsilon^2 |\ln(\varepsilon)| + o(\varepsilon^2 |\ln(\varepsilon)|) & \text{if } n = 6. \end{array} \right. \tag{28}$$

Moreover, it follows from [Aub76] that

$$a \varepsilon^{\frac{n-2}{4}} \leq C_\varepsilon \leq b \varepsilon^{\frac{n-2}{4}}$$

where $a, b > 0$ are independent of ε. If $p \geq 1$, standard computations made in [Aub76] show that there exist some constants $c, C > 0$ independent of ε such that

$$c \alpha_{p,\varepsilon} \leq \int_M v_\varepsilon^p \, dv_g \leq C \alpha_{p,\varepsilon} \tag{29}$$

where

$$\alpha_{p,\varepsilon} = \left| \begin{array}{lll} \varepsilon^{\frac{2n-(n-2)p}{4}} & \text{if} & p > \frac{n}{n-2}; \\ |\ln(\varepsilon)| \varepsilon^{\frac{n}{4}} & \text{if} & p = \frac{n}{n-2}; \\ \varepsilon^{\frac{(n-2)p}{4}} & \text{if} & p < \frac{n}{n-2} \end{array} \right.$$

Since the large inequality if easier to obtain, we only prove strict inequality. Assume first that $\mu_1(M,g) > 0$, that (M,g) is not locally conformally flat and that $n \geq 11$. We set,

$$u_\varepsilon = Y(v_\varepsilon)^{\frac{1}{N-2}} v_\varepsilon + \mu_1(M,g)^{\frac{1}{N-2}} v.$$

Let us derive estimates for $F\big(u_\varepsilon, \lambda v_\varepsilon + \mu v)\big)$. Let $(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$. Using (24), (26) and the equation (25) of v, we get that

$$F(u_\varepsilon, \lambda v_\varepsilon + \mu v) = \frac{\lambda^2 \int_M v_\varepsilon L_g(v_\varepsilon)\, dv_g + \mu^2 \int_M v L_g(v)\, dv_g + 2\lambda\mu \int_M v_\varepsilon L_g v\, dv_g}{\lambda^2 \int_M |u_\varepsilon|^{N-2}(\lambda v_\varepsilon + \mu v)^2\, dv_g} \left(\int_M u_\varepsilon^N\, dv_g\right)^{\frac{2}{n}}.$$

$$= \frac{\lambda^2 Y(v_\varepsilon) + \mu^2 \mu_1(M,g) + 2\lambda\mu\mu_1(M,g) \int_M |v|^{N-2} v v_\varepsilon\, dv_g}{\lambda^2 \int_M |u_\varepsilon|^{N-2} v_\varepsilon^2\, dv_g + \mu^2 \int_M |u_\varepsilon|^{N-2} v^2\, dv_g + 2\lambda\mu \int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g} \left(\int_M u_\varepsilon^N\, dv_g\right)^{\frac{2}{n}}. \tag{30}$$

Using the definition of u_ε

$$\begin{aligned}
&\lambda^2 \int_M |u_\varepsilon|^{N-2} v_\varepsilon^2\, dv_g + \mu^2 \int_M |u_\varepsilon|^{N-2} v^2\, dv_g + 2\lambda\mu \int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g \\
&\geq \lambda^2 Y(v_\varepsilon) \textstyle\int_M |v_\varepsilon|^N\, dv_g + \mu^2 \mu_1(M,g) \int_M |v|^N\, dv_g + 2\lambda\mu \int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g \\
&\qquad = \lambda^2 Y(v_\varepsilon) + \mu^2 \mu_1(M,g) + 2\lambda\mu \textstyle\int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g.
\end{aligned}$$

If $\lambda\mu \geq 0$, we have

$$2\lambda\mu \int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g \geq 2\lambda\mu\mu_1(M,g) \int_M v^{N-2} v_\varepsilon\, dv_g.$$

This implies that

$$\frac{\lambda^2 Y(v_\varepsilon) + \mu^2 \mu_1(M,g) + 2\lambda\mu\mu_1(M,g) \int_M |v|^{N-2} v v_\varepsilon\, dv_g}{\lambda^2 \int_M |u_\varepsilon|^{N-2} v_\varepsilon^2\, dv_g + \mu^2 \int_M |u_\varepsilon|^{N-2} v^2\, dv_g + 2\lambda\mu \int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g} \leq 1.$$

If $\lambda\mu < 0$ then, we write that since $N-2 \in]0,1[$,

$$|u_\varepsilon|^{N-2} \leq Y(v_\varepsilon) v_\varepsilon^{N-2} + \mu_1(M,g) v^{N-2}.$$

We obtain that

$$\begin{aligned}
&\lambda^2 \int_M |u_\varepsilon|^{N-2} v_\varepsilon^2\, dv_g + \mu^2 \int_M |u_\varepsilon|^{N-2} v^2\, dv_g + 2\lambda\mu \int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g \\
&\quad \geq \lambda^2 Y(v_\varepsilon) + \mu^2 \mu_1(M,g) - C\left(\int_M v^{N-1} v_\varepsilon\, dv_g + \int_M v_\varepsilon^{N-1} v\, dv_g\right).
\end{aligned}$$

where $C > 0$ is as in in the following a positive real number independent of ε. Together with (29), we get that

$$\lambda^2 \int_M |u_\varepsilon|^{N-2} v_\varepsilon^2\, dv_g + \mu^2 \int_M |u_\varepsilon|^{N-2} v^2\, dv_g + 2\lambda\mu \int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g \geq \lambda^2 Y(v_\varepsilon) + \mu^2 \mu_1(M,g) + O(\varepsilon^{\frac{n-2}{4}}).$$

It follows that

$$\sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} \frac{\lambda^2 Y(v_\varepsilon) + \mu^2 \mu_1(M,g) + 2\lambda\mu\mu_1(M,g) \int_M |v|^{N-2} v v_\varepsilon\, dv_g}{\lambda^2 \int_M |u_\varepsilon|^{N-2} v_\varepsilon^2\, dv_g + \mu^2 \int_M |u_\varepsilon|^{N-2} v^2\, dv_g + 2\lambda\mu \int_M |u_\varepsilon|^{N-2} v v_\varepsilon\, dv_g} \leq 1 + O(\varepsilon^{\frac{n-2}{4}}). \tag{31}$$

By Lemma 5.7,

$$\begin{aligned}
\int_M u_\varepsilon^N\, dv_g \quad &\leq \quad (Y(u_\varepsilon))^{\frac{n}{2}} \int_M v_\varepsilon^N\, dv_g + \mu_1(M,g)^{\frac{n}{2}} \int_M v^N\, dv_g \\
&\qquad + C\left(\int_M v^{N-1} v_\varepsilon\, dv_g + \int_M v_\varepsilon^{N-1} v\, dv_g\right).
\end{aligned}$$

By (24), (26), (28) and (29), we obtain

$$\left(\int_M u_\varepsilon^N\, dv_g\right)^{\frac{2}{n}} \leq (\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}})^{\frac{2}{n}} - C\varepsilon^2 + O(\varepsilon^{\frac{n-2}{4}}) + o(\varepsilon^2). \tag{32}$$

Since $\frac{n-2}{4} > 2$, we get from (31) and (32) that for ε small enough

$$\begin{aligned}\mu_2(M,g) &\le \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} F(u_\varepsilon, \lambda v_\varepsilon + \mu v)\\ &\le (\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}})^{\frac{2}{n}} - C\varepsilon^2 + O(\varepsilon^{\frac{n-2}{4}}) + o(\varepsilon^2) < (\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}})^{\frac{2}{n}}.\end{aligned}$$

This proves Theorem 5.4 if $\mu_1(M,g) > 0$.

Now, we assume that $\mu_1(M,g) = 0$, that (M,g) is not locally conformally flat and that $n \ge 9$. For more simplicity, We set $u_\varepsilon = v_\varepsilon$ instead of $u_\varepsilon = Y(v_\varepsilon)^{\frac{n-2}{4}} v_\varepsilon$ as above. We proceed exactly as in the case $\mu_1(M,g) > 0$. We obtain that for $(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$$\begin{aligned}F(u_\varepsilon, \lambda v_\varepsilon + \mu v) &= \frac{\lambda^2 Y(v_\varepsilon)}{\lambda^2 \int_M v_\varepsilon^N\, dv_g + \mu^2 \int_M |v_\varepsilon|^{N-2} v^2\, dv_g + 2\lambda\mu \int_M |v_\varepsilon|^{N-1} v\, dv_g}\left(\int_M v_\varepsilon^N\, dv_g\right)^{\frac{2}{n}}\\ &= \frac{\lambda^2 Y(v_\varepsilon)}{\lambda^2 + \mu^2 \int_M |v_\varepsilon|^{N-2} v^2\, dv_g + 2\lambda\mu \int_M |v_\varepsilon|^{N-1} v\, dv_g}.\end{aligned}$$

Let $\lambda_\varepsilon, \mu_\varepsilon$ be such that $\lambda_\varepsilon^2 + \mu_\varepsilon^2 = 1$ and such that

$$F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) = \sup_{(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\}} (u_\varepsilon, \lambda v_\varepsilon + \mu v).$$

If $\lambda_\varepsilon = 0$, we obtain that $F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) = 0$ and the theorem would be proven. Then we assume that $\lambda_\varepsilon \neq 0$ and we write that

$$F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) = \frac{Y(v_\varepsilon)}{1 + 2x_\varepsilon b_\varepsilon + x_\varepsilon^2 a_\varepsilon}$$

where $x_\varepsilon = \frac{\mu_\varepsilon}{\lambda_\varepsilon}$ and where, using (29)

$$b_\varepsilon = \int_M v_\varepsilon^{N-1} v\, dv_g \sim_{\varepsilon\to 0} C\varepsilon^{\frac{n-2}{4}}$$

and

$$a_\varepsilon = \int_M v_\varepsilon^{N-2} v^2\, dv_g \sim_{\varepsilon\to 0} C\varepsilon.$$

Maximizing this expression in x_ε and using (28), we get that

$$F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) \le \frac{\mu_1(\mathbb{S}^n) - C(M)\varepsilon^2 + o(\varepsilon^2)}{1 - \frac{b_\varepsilon^2}{a_\varepsilon}} = \frac{\mu_1(\mathbb{S}^n) - C(M)\varepsilon^2 + o(\varepsilon^2)}{1 - O(\varepsilon^{\frac{n-4}{2}})}.$$

Since $n \ge 9$, $\frac{n-4}{2} > 2$ and we get that for ε small,

$$F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) < \mu_1(\mathbb{S}^n).$$

This proves Theorem 5.4.

6. Existence of a minimum of $\mu_2(M,g)$

The aim of this section is to prove Theorem 1.4.

We study a sequence of metrics $(g_m)_m = (u_m^{N-2} g)_m$ ($u_m > 0$, $u_m \in C^\infty(M)$) which minimizes the infimum in the definition of $\mu_2(M,g)$ i.e. a sequence of metrics such that

$$\lim_m \lambda_2(g_m)\mathrm{Vol}(M,g_m)^{\frac{2}{n}} = \mu_2(M,g).$$

Without loss of generality, we may assume that $\mathrm{Vol}(M,g_m) = 1$ i.e. that

$$\int_M u_m^N\, dv_g = 1. \tag{33}$$

In particular, the sequence $(u_m)_m$ is bounded in $L^N(M)$ and there exists $u \in L^N(M)$, $u \geq 0$ such that $u_m \rightharpoonup u$ weakly in $L^N(M)$. We are going to prove that $u \neq 0$ and that the generalized metric $u^{N-2}g$ minimizes $\mu_2(M,g)$. Proposition 3.2 implies the existence of $v_m, w_m \in C^\infty(M)$, $v_m \geq 0$ such that

$$L_g v_m = \lambda_{1,m} u_m^{N-2} v_m \tag{34}$$

and

$$L_g w_m = \lambda_{2,m} u_m^{N-2} w_m. \tag{35}$$

where $\lambda_{i,m} = \lambda_i(g_m)$ and such that

$$\int_M u_m^{N-2} v_m^2 \, dv_g = \int_M u_m^{N-2} w_m^2 \, dv_g = 1 \text{ and } \int_M u_m^{N-2} v_m w_m \, dv_g = 0 \tag{36}$$

With these notations and by (33),

$$\lim_m \lambda_{2,m} = \mu_2(M,g).$$

Moreover, by the maximum principle, $v_m > 0$. If $\lambda_{1,m} = \lambda_{2,m}$ then w_m would be a minimizer of the functional associated to $\lambda_{1,m}$ and by the maximum principle, we would get that $w_m > 0$. This contradicts (36). Hence, $\lambda_{1,m} < \lambda_{2,m}$ for all m. The sequences $(v_m)_m$ and $(w_m)_m$ are bounded in $H_1^2(M)$. We can find $v, w \in H_1^2(M)$, $v \geq 0$ such that v_m (resp. w_m) tends to v (resp. w) weakly in $H_1^2(M)$. Together with the weak convergence of the $(u_m)_m$ towards u in $L^N(M)$, we get that in the sense of distributions

$$L_g v = \widehat{\mu}_1 u^{N-2} v \tag{37}$$

and

$$L_g w = \mu_2(M,g)\, u^{N-2} w. \tag{38}$$

where $\widehat{\mu}_1 = \lim_m \lambda_{1,m} \leq \mu_2(M,g)$.

From what we know until now, it is not clear whether v and w are linearly independent, and even if they are, their restrictions to the set $M \setminus u^{-1}(0)$ might be linearly dependent.

It will take a certain effort to prove the following claim.

Claim 6.1. *The functions $u^{\frac{N-2}{2}} v$ and $u^{\frac{N-2}{2}} w$ are linearly independent.*

Once the claim is proved, we have $\mathrm{span}(v,w) \in \mathrm{Gr}_2^u(H_1^2(M))$, and this implies that

$$\sup_{(\lambda,\mu)\neq(0,0)} F(u, \lambda v + \mu w) = \mu_2(M,g).$$

Hence, by equations (37) and (38), the generalized metric $u^{N-2}g$ minimizes $\mu_2(M,g)$, i.e. Theorem 1.4 is proved.

The first step in the proof of the claim is an estimate that avoids concentration of w_m and v_m.

Step 1. *Let $x \in M$ and $\varepsilon \in \left]0, \frac{N-2}{2}\right[$. We choose a cut-off function $\eta \in C^\infty$ such that $0 \leq \eta \leq 1$, $\eta(B_x(\delta)) \equiv 1$ (where $\delta > 0$ is a small number) and $\eta(M \setminus B_x(2\delta)) \equiv 0$, $|\nabla \eta| \leq 2/\delta$. We define $W_m = \eta |w_m|^\varepsilon w_m$. Then, we have*

$$\left(\int_M |W_m|^N \, dv_g\right)^{\frac{2}{N}} \leq \mu_2(M,g)(1-\alpha_\varepsilon)^{-1}\mu_1(\mathbb{S}^n)^{-1}\left(\int_{B_x(2\delta)} u_m^N\right)^{\frac{2}{n}}\left(\int_M |W_m|^N \, dv_g\right)^{\frac{2}{N}} + C_\delta. \tag{39}$$

where C_δ is a constant that may depend on δ but not on ε and where $\lim_{\varepsilon\to 0}\alpha_\varepsilon = 0$. Moreover, the same conclusion is true with $V_m = \eta|v_m|^\varepsilon v_m$ instead of W_m.

The proof uses classical methods. We will explain the proof for W_m. The proof for V_m uses exactly the same arguments.

At first, we differentiate the definition of W and obtain

$$\begin{aligned}
|\nabla W_m|^2 &\geq \left|\nabla(|w_m|^\varepsilon w_m)\right|^2\eta^2 - \left(2|\nabla\eta|\,|w_m|^{1+\varepsilon}\right)\left(\left|\nabla(|w_m|^\varepsilon w_m)\right|\eta\right) + |\nabla\eta|^2\,|w_m|^{2+2\varepsilon}\\
&\geq \left|\nabla\big(|w_m|^\varepsilon w_m\big)\right|^2\eta^2 - \left(\frac{1}{2}\left|\nabla(|w_m|^\varepsilon w_m)|^2\eta^2 + 2|\nabla\eta|^2|w_m|^{2+2\varepsilon}\right) + |\nabla\eta|^2|w_m|^{2+2\varepsilon}
\end{aligned}$$

This leads to

$$\eta^2|\nabla(|w_m|^\varepsilon w_m)|^2 \leq 2|\nabla W_m|^2 + 2|\nabla\eta|^2|w_m|^{2+2\varepsilon}. \tag{40}$$

Now, we want to derive lower bound for

$$(\nabla(\eta^2|w_m|^{2\varepsilon}w_m), \nabla w_m) = |\nabla W_m|^2 - \left|\nabla(\eta|w_m|^\varepsilon)\right|^2|w_m|^2 \tag{41}$$

For the second summand on the right hand side in (41) we have the bound

$$\begin{aligned}
|\nabla(\eta|w_m|^\varepsilon)|^2|w_m|^2 &= |\nabla\eta|^2|w_m|^{2+2\varepsilon} + 2(\nabla\eta, \nabla|w_m|^\varepsilon)\,\eta|w_m|^{2+\varepsilon} + \eta^2\left|\nabla(|w_m|^\varepsilon)\right|^2 w_m^2\\
&\leq 2|\nabla\eta|^2|w_m|^{2+2\varepsilon} + 2\eta^2\left|\nabla(|w_m|^\varepsilon)\right|^2 w_m^2\\
&\leq 2|\nabla\eta|^2|w_m|^{2+2\varepsilon} + \frac{2\eta^2\varepsilon^2}{(1+\varepsilon)^2}\left|\nabla(|w_m|^\varepsilon w_m)\right|^2\\
&\leq (2 + \frac{4\varepsilon^2}{(1+\varepsilon)^2})|\nabla\eta|^2|w_m|^{2+2\varepsilon} + \frac{4\varepsilon^2}{(1+\varepsilon)^2}|\nabla W_m|^2.
\end{aligned}$$

Here, we used (40) in the last line. Coming back to (41), we obtain that

$$(\nabla(\eta^2|w_m|^{2\varepsilon}w_m), \nabla w_m) \geq (1-\alpha_\varepsilon)|\nabla W_m|^2 - C|\nabla\eta|^2|w_m|^{2+2\varepsilon}.$$

where $\alpha_\varepsilon \to 0$ when $\varepsilon \to 0$ and where $C > 0$ is a constant independent of ε. This relations shows that

$$\int_M \eta^2|w_m|^{2\varepsilon}w_m L_g(w_m)\,dv_g \geq (1-\alpha_\varepsilon)\int_M c_n|\nabla W_m|^2\,dv_g - C\int_M |\nabla\eta|^2|w_m|^{2+2\varepsilon}\,dv_g + \min \mathrm{Scal}\int W_m^2\,dv_g.$$

Now, since $\varepsilon < \frac{N-2}{2}$,the sequence $(w_m)_m$ is bounded in $L^{2+2\varepsilon}(M)$ (and hence the sequence $(W_m)_m$ is bounded in $L^2(M)$). As a consequence, there exists a constant C_δ possibly depending on δ but not on ε, and such that

$$\int_M \eta^2|w_m|^{2\varepsilon}w_m L_g(w_m)\,dv_g \geq (1-\alpha_\varepsilon)\int_M \left(c_n|\nabla W_m|^2 + B_0(M,g)W_m^2\right)\,dv_g - C_\delta. \tag{42}$$

Using equation (35) in the left hand side of (42) and applying Sobolev inequality (S) to the right hand side, we get that

$$\mu_2(M,g)\int_M u_m^{N-2}W_m^2\,dv_g \geq (1-\alpha_\varepsilon)\mu_1(\mathbb{S}^n)\left(\int_M |W_m|^N\,dv_g\right)^{\frac{2}{N}} - C_\delta.$$

By the Hölder inequality, we obtain

$$\left(\int_M |W_m|^N\,dv_g\right)^{\frac{2}{N}} \leq \mu_2(M,g)(1-\alpha_\varepsilon)^{-1}\mu_1(\mathbb{S}^n)^{-1}\left(\int_{B_x(2\delta)} u_m^N\right)^{\frac{2}{n}}\left(\int_M |W_m|^N\,dv_g\right)^{\frac{2}{N}} + C_\delta.$$

This ends the proof of the step.

Step 2. *If $\mu_2(M,g) < \mu_1(\mathbb{S}^n)$, then the generalized metric $u^{N-2}g$ minimizes $\mu_2(M,g)$.*

From (39), and the fact $\mu_2(M,g) < \mu_1(\mathbb{S}^n)$, we get that for ε small enough, there exists a constant $K < 1$ such that

$$\left(\int_M |W_m|^N \, dv_g\right)^{\frac{2}{N}} \leq K \left(\int_{B_x(2\delta)} u_m^N\right)^{\frac{2}{n}} \left(\int_M |W_m|^N \, dv_g\right)^{\frac{2}{N}} + C_\delta.$$

Since $\int_{B_x(2\delta)} u_m^N \leq 1$, the sequence $\int_M |W_m|^N \, dv_g$ is bounded. This implies that $(w_m)_m$ is bounded in $L^{N+\varepsilon}(B_x(\delta))$ and since x is arbitrary in $L^{N+\varepsilon}(M)$. Weak convergences $w_m \to w$ in $H_1^2(M)$ implies strong convergence $w_m \to w$ in $L^{N-\varepsilon}(M)$. The Hölder inequality yields then strong convergence in $L^N(M)$. After passing to a subsequence we obtain that $(w_m)_m$ tends to w strongly in $L^N(M)$. This implies that we can pass to the limit in (36) and hence that $u^{\frac{N-2}{2}} v$ and $u^{\frac{N-2}{2}} w$ are linearly independent. The claim follows in this case.

In the following, we assume that $\mu_1(M,g) > 0$ and that

$$\mu_2(M,g) < \left(\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}}\right)^{\frac{2}{n}}.$$

We define the *set of concentration points*

$$\Omega = \left\{ x \in M \,\middle|\, \forall \delta > 0, \ \limsup_m \int_{B_x(\delta)} u_m^N \, dv_g > \frac{1}{2} \right\}.$$

Since $\int_M u_m^N \, dv_g = 1$, we can assume — after passing to a subsequence — that Ω contains at most one point.

We now prove that:

Step 3. *Let U be an open set such that $\overline{U} \subset M \setminus \Omega$. Then, the sequence $(v_m)_m$ (and $(w_m)_m$ resp.) converges towards v (and w resp.) strongly in $H_1^2(\overline{U})$.*

Without loss of generality, we prove the result only for w. For any $x \in M \setminus \Omega$ we can find $\delta > 0$ with

$$\limsup_m \int_{B_x(2\delta)} u_m^N \, dv_g \leq \frac{1}{2}.$$

Using $\mu_2(M,g) < (\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}})^{\frac{2}{n}} \leq 2^{\frac{2}{n}} \mu_1(\mathbb{S}^n)$ we obtain for a small $\varepsilon > 0$

$$\mu_2(M,g)(1-\alpha_\varepsilon)^{-1} \mu_1(\mathbb{S}^n)^{-1} \left(\int_{B_x(2\delta)} u_m^N\right)^{\frac{2}{n}} \leq K < 1$$

for almost all m. Together with inequality (39), this proves that $\int_M |W_m|^N \, dv_g$ is bounded. This implies that $(w_m)_m$ is bounded in $L^{N+\varepsilon}(B_x(\delta))$. As in last step, this proves that up to a subsequence, $(w_m)_m$ tends to w strongly in $L^N(U)$. Using equation (35) and (38), we easily obtain that

$$\lim_m \int_U |\nabla w_m|^2 dv_g = \int_U |\nabla w|^2 dv_g.$$

Together with the weak convergence of $(w_m)_m$ to w, this proves the step.

Now, we set for all m,

$$S_m = \{\lambda v_m + \mu w_m | \lambda^2 + \mu^2 = 1\} \ \text{ and } \ S = \{\lambda v + \mu w | \lambda^2 + \mu^2 = 1\}.$$

Step 4. *There exists a sequence $(\overline{w}_m)_m$ ($\overline{w}_m \in S_m$) and $\overline{w} \in S$ such that $\overline{w}_m$ tends to $\overline{w}$ strongly in $H_1^2(M)$.*

By theorem 4.1, there exists λ_m, μ_m such that $\lambda_m^2 + \mu_m^2 = 1$ and such that

$$\begin{aligned} 2^{2/n}\mu_1(\mathbb{S}^n) & \int_M u_m^{N-2}(\lambda_m(v_m - v) + \mu_m(w_m - w))^2 \, dv_g \\ & \leq \int_M c_n |\nabla(\lambda_m(v_m - v) + \mu_m(w_m - w))|^2 \, dv_g \\ & + \int_M B_0(M,g)(\lambda_m(v_m - v) + \mu_m(w_m - w))^2 \, dv_g \end{aligned} \tag{43}$$

Up to a subsequence, there exists λ, μ such that $\lambda^2 + \mu^2 = 1$ and such that $\lim_m \lambda_m = \lambda$ and $\lim_m \mu_m = \mu$. We set $\overline{w}_m = \lambda_m v_m + \mu_m w_m \in S_m$ and $\overline{w} = \lambda v + \mu w$. Then, $\overline{w}_m$ tends to $\overline{w}$ weakly in $H_1^2(M)$. A first remark is that by strong convergence in $L^2(M)$

$$\lim_m \int_M (\lambda_m(v_m - v) + \mu_m(w_m - w))^2 \, dv_g = 0. \tag{44}$$

Using the weak convergence of $\overline{w}_n$ to $\overline{w}$ in $H_1^2(M)$ and the weak convergence of u_m to u in $L^N(M)$, it is easy to compute that

$$\int_M u_m^{N-2}(\lambda_m(v_m - v) + \mu_m(w_m - w))^2 \, dv_g = \int_M u_m^{N-2}\overline{w}_m^2 \, dv_g - \int_M u^{N-2}\overline{w}^2 \, dv_g + o(1) \tag{45}$$

and that

$$\begin{aligned} \int_M c_n |\nabla(\lambda_m(v_m - v) + \mu_m(w_m - w))|^2 \, dv_g &= \lambda^2 \left(\int_M c_n |\nabla v_m|^2 \, dv_g - \int_M c_n |\nabla v|^2 \, dv_g \right) \\ &+ \mu^2 \left(\int_M c_n |\nabla w_m|^2 \, dv_g - \int_M c_n |\nabla w|^2 \, dv_g \right) \\ &+ 2\lambda\mu \left(\int_M c_n (\nabla v_m, \nabla w_m) \, dv_g - \int_M c_n (\nabla v, \nabla w) \, dv_g \right) + o(1). \end{aligned}$$

Using equations (34), (35), (37) and (38), we get that

$$\begin{aligned} \int_M c_n |\nabla(\lambda_m(v_m - v) + \mu_m(w_m - w))|^2 \, dv_g &= \lambda^2 \, \widehat{\mu}_1 \left(\int_M u_m^{N-2} v_m^2 \, dv_g - \int_M u^{N-2} v^2) \, dv_g \right) \\ &+ \mu^2 \, \mu_2(M,g) \left(\int_M u_m^{N-2} w_m^2 \, dv_g - \int_M u^{N-2} w^2) \, dv_g \right) \\ &+ 2\lambda\mu \, \mu_2(M,g) \left(\int_M u_m^{N-2} v_m w_m \, dv_g - \int_M u^{N-2} vw \, dv_g \right) + o(1). \end{aligned}$$

Since $\widehat{\mu}_1 \leq \mu_2(M,g)$ and since, by weak convergence

$$\liminf_m \int_M u_m^{N-2} v_m^2 \, dv_g - \int_M u^{N-2} v^2) \, dv_g \geq 0,$$

we get that

$$\begin{aligned} \int_M c_n |\nabla(\lambda_m(v_m - v) + \mu_m(w_m - w))|^2 \, dv_g &\leq \lambda^2 \, \mu_2(M,g) \left(\int_M u_m^{N-2} v_m^2 \, dv_g - \int_M u^{N-2} v^2 \, dv_g \right) \\ &+ \mu^2 \, \mu_2(M,g) \left(\int_M u_m^{N-2} w_m^2 \, dv_g - \int_M u^{N-2} w^2 \, dv_g \right) \\ &+ 2\lambda\mu \, \mu_2(M,g) \left(\int_M u_m^{N-2} v_m w_m \, dv_g - \int_M u^{N-2} vw \, dv_g \right), \end{aligned}$$

and hence,

$$\int_M c_n |\nabla(\lambda_m(v_m - v) + \mu_m(w_m - w))|^2 \, dv_g \leq \mu_2(M,g) \left(\int_M u_m^{N-2} \overline{w}_m^2 \, dv_g - \int_M u^{N-2} \overline{w}^2 \, dv_g \right) + o(1). \tag{46}$$

Together with (43), (44) and (45), we obtain that

$$2^{2/n}\mu_1(\mathbb{S}^n)\left(\int_M u_m^{N-2}\overline{w}_m^2\,dv_g-\int_M u^{N-2}\overline{w}^2\,dv_g\right)$$
$$\le \mu_2(M,g)\left(\int_M u_m^{N-2}\overline{w}_m^2\,dv_g-\int_M u^{N-2}\overline{w}^2\,dv_g\right)+o(1).$$

Since $\mu_2(M,g)<(\mu_1(M,g)^{\frac{n}{2}}+\mu_1(\mathbb{S}^n)^{\frac{n}{2}})^{\frac{2}{n}}\le 2^{\frac{2}{n}}\mu_1(\mathbb{S}^n)$, we get that

$$\left(\int_M u_m^{N-2}\overline{w}_m^2\,dv_g-\int_M u^{N-2}\overline{w}^2\,dv_g\right)\le K_0\left(\int_M u_m^{N-2}\overline{w}_m^2\,dv_g-\int_M u^{N-2}\overline{w}^2\,dv_g\right)+o(1)$$

where $K_0<1$. This implies that

$$1=\lim_m\int_M u_m^{N-2}\overline{w}_m^2\,dv_g=\int_M u^{N-2}\overline{w}^2\,dv_g \tag{47}$$

and hence by (46).

$$\lim_m\int_M c_n|\nabla(\lambda_m(v_m-v)+\mu_m(w_m-w))|^2\,dv_g=0.$$

The step easily follows.

As a remark, (47) implies that $u^{\frac{N-2}{2}}\overline{w}\not\equiv 0$.

Now, we set $\overline{v}_m=-\mu_m v_m+\lambda_m w_m$ and $\overline{v}=-\mu v+\lambda w$. We prove that

Step 5. *There exists $x\in M$ such that*

$$\limsup_m\int_{B_x\delta}u_m^2(\overline{v}_m-\overline{v})^2\,dv_g=1$$

for all $\delta>0$.

The sequence $(\overline{v}_m)_m$ tends to $\overline{v}$ weakly in $H_1^2(M)$. If $\Omega=\emptyset$, then we know from Step 3 that $(\overline{v}_m)_m$ tends to $\overline{v}$ strongly in $H_1^2(M)$, which implies $\int u^{N-2}\bar{v}\bar{w}=0$. Hence, in the case $\Omega=\emptyset$, the functions $u^{\frac{N-2}{2}}\bar{v}$ and $u^{\frac{N-2}{2}}\bar{w}$ are linearly independent, and the claim follows.

Hence, without loss of generality let $\Omega=\{x\}$ where x is some point of M. We assume that the claim is false, i.e. $u^{\frac{N-2}{2}}v$ and $u^{\frac{N-2}{2}}w$ are linearly dependent. As $u^{\frac{N-2}{2}}\overline{w}\not\equiv 0$, there exists $b\in\mathbb{R}$ with $u^{\frac{N-2}{2}}\overline{v}=bu^{\frac{N-2}{2}}\overline{w}$. Hence,

$$0=\int_M u^{N-2}\overline{v}^2\,dv_g+b^2\int_M u^{N-2}\overline{w}^2\,dv_g-2b\int_M u^{N-2}\overline{v}\,\overline{w}\,dv_g.$$

By strong convergence of $(\overline{w}_m)_m$ to $\overline{w}$ in $H_1^2(M)$, weak convergence of $(\overline{v}_m)_m$ to $\overline{v}$ in $H_1^2(M)$ and weak convergence of $(u_m)_m$ to u in $L^N(M)$, we have $\int_M u^{N-2}\overline{w}^2\,dv_g=1$ and $\int_M u^{N-2}\overline{v}\,\overline{w}\,dv_g=0$. We obtain $\int_M u^{N-2}\overline{v}^2\,dv_g+b^2=0$. As a consequence, $u^{\frac{N-2}{2}}\overline{v}\equiv 0$. Let now $\delta>0$. We write that

$$\begin{aligned}\int_{B_x(\delta)}u_m^{N-2}(\overline{v}_m-\overline{v})^2\,dv_g &= \int_{B_x(\delta)}u_m^{N-2}\overline{v}_m^2\,dv_g\\ &= 1-\int_{M\setminus B_x(\delta)}u_m^{N-2}\overline{v}_m^2\,dv_g.\end{aligned}$$

By step 3,

$$\lim_m\int_{M\setminus B_x(\delta)}u_m^{N-2}\overline{v}_m^2\,dv_g=\int_{M\setminus B_x(\delta)}u^{N-2}\overline{v}^2\,dv_g=0.$$

This proves the step.

Step 6. *Conclusion.*

Let $\delta > 0$ be a small fixed number. In the following, $o(1)$ denotes a sequence of real numbers which tends to 0, however we do not claim that the convergence is uniform in δ. By step 5 and the Hölder inequality,

$$\begin{aligned} 1 &= \int_{B_x(\delta)} u_m^{N-2}(\overline{v}_m - \overline{v})^2 \, dv_g + o(1) \\ &\leq \left(\int_{B_x(\delta)} u_m^N \, dv_g \right)^{\frac{2}{n}} \left(\int_M |\overline{v}_m - \overline{v}|^N \, dv_g \right)^{\frac{2}{n}} + o(1). \end{aligned}$$

Applying Sobolev inequality (S), we get that

$$1 \leq \left(\int_{B_x(\delta)} u_m^N \, dv_g \right)^{\frac{2}{n}} \mu_1(\mathbb{S}^n)^{-1} \left(\int_M c_n |\nabla(\overline{v}_m - \overline{v})|^2 \, dv_g + B_0(M,g) \int_M (\overline{v}_m - \overline{v})^2 \, dv_g \right) + o(1).$$

By strong convergence of $(\overline{v}_m - \overline{v})_m$ to 0 in $L^2(M)$,

$$1 \leq \left(\int_{B_x(\delta)} u_m^N \, dv_g \right)^{\frac{2}{n}} \mu_1(\mathbb{S}^n)^{-1} \left(\int_M c_n |\nabla(\overline{v}_m - \overline{v})|^2 + S_g (\overline{v}_m - \overline{v})^2 \, dv_g \right) + o(1)$$

Using equations (34), (35), (37), (38) and the fact that $\widehat{\mu}_1 \leq \mu_2(M,g)$, we get that

$$\begin{aligned} 1 &\leq \left(\int_{B_x(\delta)} u_m^N \, dv_g \right)^{\frac{2}{n}} \mu_1(\mathbb{S}^n)^{-1} \mu_2(M,g) \int_M u_m^{N-2} (\overline{v}_m - \overline{v})^2 \, dv_g \\ &= \left(\int_{B_x(\delta)} u_m^N \, dv_g \right)^{\frac{2}{n}} \mu_1(\mathbb{S}^n)^{-1} \mu_2(M,g). \end{aligned}$$

Since $\mu_2(M,g) < (\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}})^{\frac{2}{n}}$, we obtain that

$$\int_{B_x(\delta)} u_m^N \, dv_g > \frac{\mu_1(\mathbb{S}^n)^{\frac{n}{2}}}{\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}}}.$$

and since $\int_M u_m^N \, dv_g = 1$,

$$\int_{M \setminus B_x(\delta)} u_m^N \, dv_g < \frac{\mu_1(M,g)^{\frac{n}{2}}}{\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}}}. \tag{48}$$

Now, we write that by strong convergence of $(\overline{w}_m)_m$ in $H_1^2(M)$,

$$a_\delta = \int_{B_x(\delta)} u_m^{N-2} \overline{w}_m^2 \, dv_g$$

$$1 - a_\delta = \int_{M \setminus B_x(\delta)} u_m^{N-2} \overline{w}_m^2 \, dv_g$$

where a_δ does not depend of m and tends to 0 when δ tends to 0. By Hölder inequality,

$$1 - a_\delta \leq \left(\int_{M \setminus B_x(\delta)} u_m^N \, dv_g \right)^{\frac{2}{n}} \left(\int_M \overline{w}^N \, dv_g \right)^{\frac{2}{n}}.$$

Since $\mu_1(M,g)$ is the minimum of Yamabe functional, we get that

$$1 - a_\delta \leq \left(\int_{M \setminus B_x(\delta)} u_m^N \, dv_g \right)^{\frac{2}{n}} \mu_1(M,g)^{-1} \int_M \left(c_n |\nabla \overline{w}_m|^2 + S_g \overline{w}_m^2 \right) \, dv_g.$$

As we did for $\overline{v}$, we obtain

$$1 - a_\delta \quad \leq \quad \left(\int_{M \backslash B_x(\delta)} u_m^N \, dv_g \right)^{\frac{2}{n}} \mu_1(M,g)^{-1} \mu_2(M,g) \underbrace{\int_M u_m^{N-2} \overline{w}_m^2 \, dv_g}_{1}$$

By (48), in the limit $\delta \to 0$, this gives

$$\mu_2(M,g) \geq (\mu_1(M,g)^{\frac{n}{2}} + \mu_1(\mathbb{S}^n)^{\frac{n}{2}})^{\frac{2}{n}}.$$

This is false by assumption. Hence, the claim is proved, and Theorem 1.4 follows.

7. The invariant $\mu_k(M)$ for $k \geq 3$

A natural question is: Can we do the same work for $\mu_k(M)$ with $k \geq 3$? This problem is still open but seems to be hard. Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Using the variational characterization of $\mu_k(M)$, one can check that $\mu_k(M) \leq k^{\frac{2}{n}} \mu_1(\mathbb{S}^n)$. It is natural to conjecture that one has equality if M is the round sphere i.e. that $\mu_k(\mathbb{S}^n) = k^{\frac{2}{n}} \mu_1(\mathbb{S}^n)$. However, the following result shows that is false:

Proposition 7.1. *Let $n \in \mathbb{N}^*$. Then, for $n \geq 7$*

$$\mu_{n+2}(\mathbb{S}^n) < (n+2)^{\frac{2}{n}} \mu_1(\mathbb{S}^n).$$

Proof: Let us study $\mathbb{S}^n$ with its natural embedding into $\mathbb{R}^{n+1}$. We have $L_g(1) = n(n-1)$. Hence, $\lambda_1(\mathbb{S}^n) \leq n(n-1)$. Let also x_i ($i \in [1, \cdots, n+1]$) be the canonical coordinates on $\mathbb{R}^{n+1}$. As one can check,

$$L_g(x_i) = \frac{n(n-1)(n+2)}{n-2} x_i$$

and hence $\lambda_{n+2}(\mathbb{S}^n) \leq \frac{n(n-1)(n+2)}{n-2}$. This shows that

$$\mu_{n+2}(\mathbb{S}^n) \leq \frac{n(n-1)(n+2)}{n-2} \omega_n^{\frac{2}{n}}.$$

As one can check, for $n \geq 7$

$$\frac{n(n-1)(n+2)}{n-2} \omega_n^{\frac{2}{n}} < (n+2)^{\frac{2}{n}} n(n-1) \omega_n^{\frac{2}{n}} = (n+2)^{\frac{2}{n}} \mu_1(\mathbb{S}^n).$$

This ends the proof of Proposition 7.1.

8. The case of manifolds whose Yamabe invariant is negative

We let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$. Then, we have:

Proposition 8.1. *Let $k \in \mathbb{N}^*$. Assume that $\mu_k(M,g) < 0$. Then, $\mu_k(M,g) = -\infty$.*

Proof: After a possible change of metric in the conformal class, we can assume that $\lambda_k(g) < 0$. This implies that we can find some smooth functions $v_1, \cdots, v_k$ satisfying

$$L_g v_i = \lambda_i(g) v_i$$

for all $i \in \{1, \cdots, k\}$ and such that

$$\int_M v_i v_j dv_g = 0$$

for all $i, j \in \{1, \cdots, k\}$, $i \neq j$. Let v_ε be defined as in the proof of Theorem 5.4. We define $u_\varepsilon = v_\varepsilon + \varepsilon$ to obtain a positive function. We set $V = \{v_1, \cdots, v_k\}$. It is easy to check that, uniformly in $v \in V$

$$\lim_{\varepsilon to 0} \int_M v_\varepsilon^{N-2} v^2 dv_g = 0.$$

Since $\lambda_i < 0$, it is then easy to see that $\sup_{v \in V} F(v_\varepsilon, v) = -\infty$. Together with the variational characterization of $\mu_k(M,g)$, we get that $\mu_k(M,g) = -\infty$.

This result proves for example that if the Yamabe invariant of (M,g) is negative, then $\mu_1(M,g) = -\infty$. This is the reason why we restricted in this article to the case of non-negative Yamabe invariant. Many of our results and proofs remain valid in the case $\mu_2(M) \geq 0$. However, if the Yamabe invariant of (M,g) is non-positive, there are other ways to find nodal solutions of Yamabe equation. Indeed, Aubin's methods [Aub76] can be applied to avoid concentration phenomenom. See for example [DJ02], [Jou99], [Hol99] for such methods. Here, we present very briefly one new method in this case. We just sketch it since it is not the purpose of our paper to find solutions of Yamabe equation with Aubin's type methods.

At first, for any metric $\tilde{g}$ conformal to g, we let $\lambda_1^+(\tilde{g})$ be the first *positive* eigenvalue of Yamabe operator. We then define $\lambda^+ = \inf \lambda_1^+(\tilde{g})\mathrm{Vol}(M,\tilde{g})^{\frac{2}{n}}$ where the infimum is taken over the conformal class of g. Then, proceeding in a way analogous to [Amm03a, Amm04], one shows that

$$0 < \lambda^+ = \inf \frac{\left(\int_M |L_g u|^{\frac{2n}{n+2}}\, dv_g\right)^{\frac{n+2}{n}}}{\int_M u L_g u \, dv_g}$$

where the infimum is taken over the smooth functions u such that

$$\int_M u L_g u \, dv_g > 0.$$

Then, one shows using test functions that $\lambda_+ \leq \mu_1(\mathbb{S}^n)$. If the inequality is strict, then we can find a minimizer for the functional above which is a solution of the Yamabe equation. If the Yamabe invariant is positive, this solution is a Yamabe metric and hence is positive. However, if the Yamabe invariant is non-positive, this solution has an alternating sign.

A. Appendix: Proof of Lemma 3.1

Let (M,g) be a compact Riemannian manifold of dimension $n \geq 3$ and let $v \in H_1^2(M)$, $v \not\equiv 0$ and $u \in L_+^N(M)$ be two functions which satisfy in the sense of distributions

$$L_g v = u^{N-2} v. \tag{Eq}$$

We define $v_+ = \sup(v,0)$. We let $q \in]1, \frac{n}{n-2}]$ be a fixed number and $l > 0$ be a large real number which will tend to $+\infty$. We let $\beta = 2q-1$. We then define the following functions for $x \in \mathbb{R}$:

$$G_l(x) = \left| \begin{array}{ccc} 0 & \text{if} & x < 0 \\ x^\beta & \text{if} & x \in [0,l[\\ l^{q-1}(ql^{q-1}x - (q-1)l^q) & \text{if} & x \geq l \end{array} \right.$$

and

$$F_l(x) = \left| \begin{array}{ccc} 0 & \text{if} & x < 0 \\ x^q & \text{if} & x \in [0,l[\\ ql^{q-1}x - (q-1)l^q & \text{if} & x \geq l \end{array} \right.$$

It is easy to check that for all $x \in \mathbb{R}$,

$$(F_l'(x))^2 \leq q G_l'(x), \tag{49}$$

$$(F_l(x))^2 \geq x G_l(x) \tag{50}$$

and

$$xG'(x) \leq \beta G_l(x). \tag{51}$$

Since F_l and G_l are uniformly lipschitz continuous functions, $F_l(v_+)$ and $G_l(v_+)$ belong to $H_1^2(M)$. Now, let $x_0 \in M$ be any point of M. We denote by η a C^2 non-negative function supported in $B_{x_0}(2\delta)$ ($\delta > 0$ being a small number to be fixed) such that $0 \leq \eta \leq 1$ and such that $\eta(B_{x_0}(\delta)) = \{1\}$. Multiply equation (Eq) by $\eta^2 G_l(v_+)$ and integrate over M. Since the supports of v_+ and $G_l(v_+)$ coincide, we get:

$$c_n \int_M (\nabla v_+, \nabla \eta^2 G_l(v_+)) dv_g + \int_M S_g v_+ \eta^2 G_l(v_+) dv_g = \int_M u^{N-2} v_+ \eta^2 G_l(v_+) dv_g. \tag{52}$$

Let us deal with the first term of the left hand side of (52). In the following, C will denote a positive constant depending possibly on η, q, β, δ but not on l. We have

$$\begin{aligned}
\int_M (\nabla v_+, \nabla \eta^2 G_l(v_+)) dv_g &= \int_M G_l(v_+)(\nabla v_+, \nabla \eta^2) dv_g + \int_M G_l'(v_+)\eta^2 |\nabla v_+|^2 dv_g \\
&= \int_M G_l(v_+) v_+ \Delta(\eta^2) - 2\int_M v_+ G_l'(v_+)\eta(\nabla v_+, \nabla \eta) dv_g + \int_M G_l'(v_+)\eta^2 |\nabla v_+|^2 dv_g \\
&\geq -C \int_M v_+ G_l(v_+) dv_g - 2\int_M v_+^2 G_l'(v_+)|\nabla \eta|^2 dv_g + \frac{1}{2}\int_M G_l'(v_+)\eta^2 |\nabla v_+|^2 dv_g.
\end{aligned}$$

Using (49), (50) and (51), we get

$$\begin{aligned}
\int_M (\nabla v_+, \nabla \eta^2 G_l(v_+)) dv_g &\geq -C \int_M (F_l(v_+))^2 dv_g + \frac{1}{2q}\int_M (F_l'(v_+))^2 \eta^2 |\nabla v_+|^2 dv_g \\
&\geq -C \int_M (F_l(v_+))^2 dv_g + \frac{1}{2q}\int_M \eta^2 |\nabla F_l(v_+)|^2 dv_g \\
&\geq -C \int_M (F_l(v_+))^2 dv_g + \frac{1}{4q}\int_M |\nabla(\eta F(v_+))|^2 dv_g - \frac{1}{2q}\int_M |\nabla \eta|^2 (F_l(v_+))^2 dv_g \\
&\geq -C \int_M (F_l(v_+))^2 dv_g + \frac{1}{4q}\int_M |\nabla(\eta F(v_+))|^2 dv_g. \qquad (53)
\end{aligned}$$

Using the Sobolev embedding $H_1^2(M)$ into $L^N(M)$, there exists a constant $A > 0$ depending only on (M, g) such that

$$\int_M |\nabla(\eta F(v_+))|^2 dv_g \geq A\left(\int_M (\eta F(v_+))^N dv_g\right)^{\frac{2}{N}} - \int_M (\eta F(v_+))^2 dv_g.$$

Together with (53), we obtain

$$\int_M (\nabla v_+, \nabla \eta^2 G_l(v_+)) dv_g \geq -C \int_M (F_l(v_+))^2 dv_g + \frac{A}{4q}\left(\int_M (\eta F(v_+))^N dv_g\right)^{\frac{2}{N}} \tag{54}$$

Independently, we choose $\delta > 0$ small enough such that

$$\int_{B_{x_0}(2\delta)} u^N dv_g \leq \left(c_n \frac{A}{8q}\right)^{\frac{n}{2}}.$$

Relation (50) and Hölder inequality then lead to

$$\int_M u^{N-2} v_+ \eta^2 G_l(v_+) dv_g \leq \int_M u^{N-2}\eta^2 (F_l(v_+))^2 dv_g \leq c_n \frac{A}{8q}\left(\int_M (\eta F(v_+))^N dv_g\right)^{\frac{2}{N}}. \tag{55}$$

Since, by (50),

$$\int_M S_g v_+ \eta^2 G_l(v_+) dv_g \geq -C \int_M (F_l(v_+)^2) dv_g,$$

we get from (52), (54) and (55) that

$$c_n \frac{A}{8q} \left(\int_M (\eta F(v_+))^N dv_g \right)^{\frac{2}{N}} \leq C \int_M (F_l(v_+))^2 dv_g.$$

Now, by Sobolew embedding, $v_+ \in L^N(M)$. Since $2q \leq N$ and since C does not depend on l, the right hand side of this inequality is bounded when l tends to $+\infty$. We obtain that

$$\limsup_{l \to +\infty} \int_M (\eta F(v_+))^N dv_g < +\infty.$$

This proves that $v_+ \in L^{qN}(B_{x_0}(\delta))$. Since x_0 is arbitrary, we get that $v_+ \in L^{qN}(M)$. Doing the same with $\sup(-v, 0)$ instead of v_+, we get that $v \in L^{qN}(M)$. This proves Lemma 3.1.

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